TRANSIENT MOTION OF GROUNDWATER (SUBSURFACE WATER) IN THE PRESENCE OF EVAPORATION

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The spread of a mound of groundwater in the region between two parallel channels with different water levels (H_1 at $x=0$ and H_2 at $x=L$) during irrigation is studied with due allowance for evaporation. Evaporation is taken into account in relation to the depth of the groundwater $h(x, t)$; its intensity is regarded as zero when $h < h_0$ (where h_0 is the critical level of the groundwater), while varying linearly or remaining constant when $h > h_0$. The intensity or irrigation is regarded as constant. This problem is solved by using the thermal potentials of a double layer and reduces to the solution of a nonlinear integral equation.

1. The intensity of evaporation $\varkappa(x, t)$ is a nonlinear function of the depth of the groundwater $h(x, t)$

$$\varkappa(x, t)=\begin{cases}0 & \text{for } h < h_0\\ -bh(x, t)+dH & \text{for } h > h_0.\end{cases} \tag{1.1}$$

In Eq. (1.1) either $b > 0$, $d=b$, $H=h_0$ (linear dependence of the evaporation on the depth of the groundwater) or $b=0$, $dH=-\varepsilon<0$ (constant evaporation). Thus the intensity of evaporation is denoted by ε when constant.

For simplicity we shall assume that the initial mound of groundwater $h(x, 0)=\varphi(x)$ intersects the plane $h(x, t)=h_0$ at not more than one point $x=x_0$. We shall also consider, in order to make the analysis specific, that the inequalities $H_1 \le h_0 \le H_2$; $\varphi(0) \le \varphi(x_0) \le \varphi(L)$ are satisfied.

The problem then reduces to the solution of the following problems for the functions $h_1(x, t)$ and $h_2(x, t)$:

$$\begin{gathered}\frac{\partial h_1}{\partial t}=a^2\frac{\partial^2 h_1}{\partial x^2}+\alpha,\quad h_1(0, t)=H_1,\quad \left(a^2=\frac{kH_*}{\sigma}\right);\\ h_1[\chi(t), t]=h_0\quad (0<x<\chi(t)),\\ h_1(x, 0)=\varphi(x)\quad (0<x<x_0),\end{gathered} \tag{1.2}$$

$$\begin{gathered}\frac{\partial h_2}{\partial t}=a^2\frac{\partial^2 h_2}{\partial x^2}+\alpha-bh_2+dH,\quad h_2[\chi(t), t]=h_0,\\ h_2(L, t)=H_2\quad (\chi(t)<x<L).\\ h_2(x, 0)=\varphi(x)\quad (x_0<x<L).\end{gathered} \tag{1.3}$$

In Eq. (1.2) k is the filtration coefficient, σ is the deficiency of saturation or water delivery; H_* is a certain average level of the groundwater; H_1 and H_2 are the water levels in the channels; L is the distance between the channels; α is the intensity of irrigation. In Eqs. (1.2) and (1.3) $x=\chi(t)$ is the equation of the boundary (moving with time and not known in advance) at which the level of the groundwater is equal to the critical value $h[\chi(t), t]=h_0$ and the flow continuity condition is satisfied

$$\frac{\partial h_1[\chi(t), t]}{\partial x}=\frac{\partial h_2[\chi(t), t]}{\partial x}, \tag{1.4}$$

where $h_1(x, t)$ is the depth of the groundwater in the region $0 \le x \le \chi$; $h_2(x, t)$ in the region $\chi(t) \le x \le L$.

We note that problems similar in presentation to (1.2)-(1.4) were solved in [1-5].

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Clearly the equation $\chi(0) = x_0$.

Let us put

$$h_1 = -\frac{\alpha}{2a^2}x^2 + h_0 + u(x, t). \tag{1.5}$$

Problem (1.2) then reduces to finding the solution of the following:

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2};\ u(0, t) = H_1 - h_0; \tag{1.6}$$

$$u[\chi(t), t] = \frac{\alpha}{2a^2}\chi^2(t);\ (0 < x < \chi(t));$$

$$u(x, 0) = \psi(x) = \varphi(x) - h_0 + \frac{\alpha}{2a^2}x^2,\ (0 < x < x_0).$$

Regarding $\chi(t)$ as a known, differentiable function, we write the solution to this problem in the form [6]

$$u(x, t) = F(x, t) + \int_0^t \frac{x\exp\left[-\frac{x^2}{4a^2(t-\tau)}\right]v_1(\tau)\,d\tau}{(t-\tau)^{3/2}} + \int_0^t \frac{[x-\chi(\tau)]\exp\left\{-\frac{[x-\chi(\tau)]^2}{4a^2(t-\tau)}\right\}v_2(\tau)\,d\tau}{(t-\tau)^{3/2}}; \tag{1.7}$$

$$F(x, t) = \int_0^{x_0}\frac{\psi(\xi)\exp\left[-\frac{(x-\xi)^2}{4a^2t}\right]d\xi}{2a\sqrt{\pi t}}.$$

If in the $b \neq 0$ case we put

$$h_2(x, t) = h_0 + \frac{\alpha}{b} + \exp(-bt)\,w_1(x, t), \tag{1.8}$$

we reduce problem (1.3) to the following:

$$\frac{\partial w_1}{\partial t} = a^2\frac{\partial^2 w_1}{\partial x^2};\quad w_1[\chi(t), t] = -\frac{\alpha}{b}\exp(bt); \tag{1.9}$$

$$w_1(L, t) = \left(H_2 - h_0 - \frac{\alpha}{b}\right)\exp(bt),\quad (\chi(t) < x < L);$$

$$w_1(x, 0) = \omega_1(x) = \varphi(x) - h_0 - \frac{\alpha}{b},\ (x_0 < x < L).$$

If $b = 0$ we put

$$h_2(x, t) = -\frac{(\alpha-\varepsilon)}{2a^2}x^2 + h_0 + w_2(\chi, t). \tag{1.10}$$

From (1.3) we obtain

$$\frac{\partial w_2}{\partial t} = a^2\frac{\partial^2 w_2}{\partial x^2},\quad w_2[\chi(t), t] = \frac{\alpha-\varepsilon}{2a^2}\chi^2(t); \tag{1.11}$$

$$w_2(L, t) = H_2 - h_0 + \frac{\alpha-\varepsilon}{2a^2}L^2,\quad (\chi(t) < x < L);$$

$$w_2(x, 0) = \omega_2(x) = \varphi(x) + \frac{\alpha-\varepsilon}{2a^2}x^2 - h_0,\ (x_0 < x < L).$$

The solution to problems (1.9)-(1.11) may be written [6]

$$w_i(x, t) = \Phi_i(x, t) + \int_0^t \frac{[x-\chi(\tau)]\exp\left\{-\frac{[x-\chi(\tau)]^2}{4a^2(t-\tau)}\right\}v_{3i}(\tau)\,d\tau}{(t-\tau)^{3/2}} + \int_0^t \frac{(x-L)\exp\left[-\frac{(x-L)^2}{4a^2(t-\tau)}\right]v_{4i}(\tau)\,d\tau}{(t-\tau)^{3/2}}; \tag{1.12}$$

$$\Phi_i(x, t) = \int_{x_0}^{L}\frac{w_i(\xi)\exp\left[-\frac{(x-\xi)^2}{4a^2t}\right]d\xi}{2a\sqrt{\pi t}},\quad (i = 1, 2).$$

In Eqs. (1.7) and (1.12) $v_1(\tau)$, $v_2(\tau)$, $v_{3i}(\tau)$, $v_{4i}(\tau)$ $(i=1, 2)$ are unknown functions; they are determined from the boundary conditions (1.6), (1.9), (1.11), which, in accordance with [6], gives the system of equations

$$v_1(t) = \nu_1(t) + \int_0^t P(t,\tau)\, v_2(\tau)\, d\tau; \tag{1.13}$$

$$v_2(t) = \nu_2(t) + \int_0^t R(t,\tau)\, v_2(\tau)\, d\tau + \int_0^t S(t,\tau)\, v_1(\tau)\, d\tau;$$

$$v_{3i}(t) = \nu_{3i}(t) - \int_0^t R(t,\tau)\, v_{3i}(\tau)\, d\tau + \int_0^t U(t,\tau)\, v_{4i}(\tau)\, d\tau;$$

$$v_{4i}(t) = \nu_{4i}(t) + \int_0^t M(t,\tau)\, v_{3i}(\tau)\, d\tau, \quad (i = 1, 2).$$

Here we have introduced the following notation:

$$P(t,\tau) = \frac{c\chi(\tau)}{(t-\tau)^{3/2}} \exp\left[-\frac{\chi^2(\tau)}{4a^2(t-\tau)}\right]; \tag{1.14}$$

$$R(t,\tau) = c\,\frac{[\chi(t)-\chi(\tau)]}{(t-\tau)^{3/2}} \exp\left\{-\frac{[\chi(t)-\chi(\tau)]^2}{4a^2(t-\tau)}\right\};$$

$$S(t,\tau) = \frac{c\chi(t)}{(t-\tau)^{3/2}} \exp\left[-\frac{\chi^2(t)}{4a^2(t-\tau)}\right];$$

$$U(t,\tau) = \frac{c[L-\chi(t)]}{(t-\tau)^{3/2}} \exp\left\{-\frac{[\chi(t)-L]^2}{4a^2(t-\tau)}\right\};$$

$$M(t,\tau) = \frac{c[L-\chi(\tau)]}{(t-\tau)^{3/2}} \exp\left\{-\frac{[\chi(\tau)-L]^2}{4a^2(t-\tau)}\right\};$$

$$\nu_1(t) = c[H_1 - h_0 - F(0,t)]; \quad \nu_2(t) = c\left\{-\frac{\alpha}{2a^2}\chi^2(t) + F[\chi(t),t]\right\};$$

$$\nu_{31}(t) = c\left\{-\frac{\alpha}{b}\exp(bt) - \Phi_1[\chi(t),t]\right\},$$

$$\nu_{41}(t) = -c\left\{\left(H_2 - h_0 - \frac{\alpha}{b}\right)\exp(bt) - \Phi_1(L,t)\right\};$$

$$\nu_{32}(t) = c\left\{\frac{\alpha-\varepsilon}{2a^2}\chi^2(t) - \Phi_2[\chi(t),t]\right\};$$

$$\nu_{42}(t) = -c\left\{H_2 - h_0 + \frac{\alpha-\varepsilon}{2a^2}L^2 - \Phi_2(L,t)\right\};$$

$$(c = [2a\sqrt{\pi}]^{-1}).$$

Substituting the function $v_1(t)$ from the first equation of the system (1.13) into the second and using the Dirichlet equation, for the function $v_2(t)$ we obtain a linear Volterra integral equation of the second kind with a singular kernel of the type $K(t,\tau) = G(t,\tau)/\sqrt{t-\tau}$ [7], where $G(t,\tau)$ is a regular function

$$v_2(t) = f(t) + \int_0^t Q(t,\tau)\, v_2(\tau)\, d\tau. \tag{1.15}$$

Here we have used the notation

$$Q(t,\tau) = R(t,\tau) + c^2 \int_\tau^t \frac{\chi(t)-\chi(\tau)}{[(t-\sigma)(\sigma-\tau)]^{3/2}} \exp\left\{-\frac{1}{4a^2}\left[\frac{\chi^2(t)}{t-\sigma} + \frac{\chi^2(\tau)}{\sigma-\tau}\right]\right\} d\sigma;$$

$$f(t) = \nu_2(t) + \int_0^t S(t,\tau)\, \nu_1(\tau)\, d\tau.$$

In an analogous way for the functions $v_{3i}(t)$ we obtain an integral equation of the same type

$$v_{3i}(t) = \varphi_i(t) + \int_0^t W(t,\tau)\, v_{3i}(\tau)\, d\tau; \tag{1.16}$$

$$W(t,\tau) = -R(t,\tau) + c^2 \int_\tau^t \frac{[L-\chi(t)][L-\chi(\tau)]}{[(t-\sigma)(\sigma-\tau)]^{3/2}} \exp\left\{-\frac{1}{4a^2}\left[\frac{\{\chi(t)-L\}^2}{t-\sigma} + \frac{\{\chi(\tau)-L\}^2}{\sigma-\tau}\right]\right\} d\sigma;$$

$$\varphi_i(t) = \nu_{3i}(t) + \int_0^t U(t,\tau)\, \nu_{4i}(\tau)\, d\tau.$$

Thus, on the assumption that the function $\chi(t)$ is known, the problem reduces to the solution of Eqs. (1.15) and (1.16), after which the functions $v_1(t)$ and $v_{4i}(t)$, $(i = 1, 2)$ are determined from Eqs. (1.13).

If we replace $v_2(\tau)$ and $v_{3i}(\tau)$ in (1.15) and (1.16) by the right-hand sides of these equations, we reduce the latter to Volterra equations with a regular kernel

$$\begin{aligned} v_2(t) &= \psi(t) + \int_0^t V_1(t,\tau)\, v_2(\tau)\, d\tau, \\ v_{3i}(t) &= \psi_i(t) + \int_0^t G_1(t,\tau)\, v_{3i}(\tau)\, d\tau, \\ \psi(t) &= f(t) + \int_0^t Q(t,\tau)\, f(\tau)\, d\tau, \\ V_1(t,\tau) &= \int_\tau^t Q(t,\sigma)\, Q(\sigma,\tau)\, d\sigma, \\ \psi_i(t) &= \varphi_i(t) + \int_0^t W(t,\tau)\, \varphi_i(\tau)\, d\tau, \\ G_1(t,\tau) &= \int_\tau^t W(t,\sigma)\, W(\sigma,\tau)\, d\sigma, \quad (i = 1, 2). \end{aligned} \tag{1.17}$$

The solution of Eqs. (1.17) takes the form

$$v_2(t) = \psi(t) + \int_0^t F(t,\tau)\, \psi(\tau)\, d\tau,$$

$$v_{3i}(t) = \psi_i(t) + \int_0^t F(t,\tau)\, \psi_i(\tau)\, d\tau,$$

where

$$F(t,\tau) = \sum_{m=1}^{\infty} V_m(t,\tau);$$

$$V_{m+1}(t,\tau) = \int_\tau^t V_1(t,\sigma)\, V_m(\sigma,\tau)\, d\sigma;$$

$$E(t,\tau) = \sum_{m=1}^{\infty} G_m(t,\tau);$$

$$G_{m+1}(t,\tau) = \int_\tau^t G_1(t,\sigma)\, G_m(\sigma,\tau)\, d\sigma,$$

in which if $|V_1(t,\tau)| \le V_0$; $|G_1(t,\tau)| \le G_0$ we have the estimates

$$|F(t,\tau)| \leqslant V_0 \exp[V_0(t-\tau)],$$
$$|E(t,\tau)| \leqslant G_0 \exp[G_0(t-\tau)].$$

It remains to find the function $\chi(t)$.

2. The function $\chi(t)$ is determined from Eq. (1.4). Making use of Eqs. (1.5) and (1.8), we obtain the condition for determining $\chi(t)$ in the $b \neq 0$ case in the form

$$\frac{\partial u[\chi(t), t]}{\partial x} = \exp(-bt)\frac{\partial w_1[\chi(t), t]}{\partial x} + \frac{\alpha}{a^2}\chi(t). \tag{2.1}$$

If $b = 0$, then by virtue of (1.10) and (1.5) this condition becomes

$$\frac{\partial u[\chi(t), t]}{\partial x} = \frac{\partial w_2[\chi(t), t]}{\partial x} + \frac{\varepsilon}{a^2}\chi(t). \tag{2.2}$$

In order to find the derivative $\partial u[\chi(t), t]/\partial x$ we transform the integral

$$I = \int_0^t \frac{[x-\chi(t)]}{(t-\tau)^{3/2}} \exp\left\{-\frac{[x-\chi(\tau)]^2}{4a^2(t-\tau)}\right\} v_2(\tau)\, d\tau,$$

in Eq. (1.7) as follows

$$I = I_1 + I_2,$$

$$I_1 = \int_0^t \frac{[x-\chi(t)]}{(t-\tau)^{3/2}} \exp\left\{-\frac{[x-\chi(t)]^2}{4a^2(t-\tau)}\right\} v_2(t)\,d\tau,$$

$$I_2 = \int_0^t \frac{N(x,t,\tau)\,d\tau}{(t-\tau)^{3/2}}, \tag{2.3}$$

$$N(x,t,\tau) = [x-\chi(\tau)]\exp\left\{-\frac{[x-\chi(\tau)]^2}{4a^2(t-\tau)}\right\} v_2(\tau) - [x-\chi(t)]\exp\left\{-\frac{[x-\chi(t)]^2}{4a^2(t-\tau)}\right\} v_2(t).$$

We then obtain

$$I_1 = -2a\sqrt{\pi}v_2(t)\left[1-\Phi\left(\frac{\chi(t)-x}{2a\sqrt{t}}\right)\right],$$

$$\left(\Phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-s^2)\,ds\right), \tag{2.4}$$

in which

$$\frac{\partial I_1}{\partial x} = -\frac{2v_2(t)}{\sqrt{t}} \exp\left\{-\frac{[\chi(t)-x]^2}{4a^2 t}\right\}. \tag{2.5}$$

Allowing for (2.3)-(2.5) and making use of Eqs. (1.7) we obtain an expression for the derivative $\partial u[\chi(t), t]/\partial x$

$$\frac{\partial u[\chi(t),t]}{\partial x} = \frac{\partial F[\chi(t),t]}{\partial x} - \frac{2v_2(t)}{\sqrt{t}} + \int_0^t \frac{\exp\left[-\frac{\chi^2(t)}{4a^2(t-\tau)}\right] v_1(\tau)\,d\tau}{(t-\tau)^{3/2}} + \int_0^t \frac{v_2(\tau)-v_2(t)}{(t-\tau)^{3/2}}\,d\tau -$$
$$-\frac{\chi^2(t)}{2a^2}\int_0^t \frac{\exp\left[-\frac{\chi^2(t)}{4a^2(t-\tau)}\right] v_1(\tau)\,d\tau}{(t-\tau)^{5/2}} + \int_0^t \frac{\left\{\exp\left\{-\frac{[\chi(t)-\chi(\tau)]^2}{4a^2(t-\tau)}\right\}-1\right\}}{(t-\tau)^{3/2}} v_2(\tau)\,d\tau -$$
$$-\frac{1}{2a^2}\int_0^t \frac{[\chi(t)-\chi(\tau)]^2}{(t-\tau)^{5/2}} \exp\left\{-\frac{[\chi(t)-\chi(\tau)]^2}{4a^2(t-\tau)}\right\} v_2(\tau)\,d\tau. \tag{2.6}$$

Using transformations analogous to (2.3)-(2.5) we derive an expression for the derivatives $\partial w_i[\chi(t), t]/\partial x$ from Eqs. (1.12)

$$\frac{\partial w_i[\chi(t),t]}{\partial x} = \frac{\partial \Phi_i[\chi(t),t]}{\partial x} - \frac{2v_{3i}(t)}{\sqrt{t}} + \int_0^t \frac{\exp\left\{-\frac{[L-\chi(t)]^2}{4a^2(t-\tau)}\right\} v_{4i}(\tau)\,d\tau}{(t-\tau)^{3/2}} + \int_0^t \frac{[v_{3i}(\tau)-v_{3i}(t)]\,d\tau}{(t-\tau)^{3/2}} -$$
$$-\frac{1}{2a^2}\int_0^t \frac{[L-\chi(t)]^2}{(t-\tau)^{5/2}} \exp\left\{-\frac{[L-\chi(t)]^2}{4a^2(t-\tau)}\right\} v_{4i}(\tau)\,d\tau + \int_0^t \frac{\left[\exp\left\{-\frac{[\chi(t)-\chi(\tau)]^2}{4a^2(t-\tau)}\right\}-1\right]}{(t-\tau)^{3/2}} v_{3i}(\tau)\,d\tau -$$
$$-\frac{1}{2a^2}\int_0^t \frac{[\chi(t)-\chi(\tau)]^2}{(t-\tau)^{5/2}} \exp\left\{-\frac{[\chi(t)-\chi(\tau)]^2}{4a^2(t-\tau)}\right\} v_{3i}(\tau)\,d\tau. \tag{2.7}$$

Substituting Eqs. (2.6) and (2.7) into (2.1) and (2.2) and multiplying the resultant equations by $\sqrt{t}$, we obtain nonlinear integral equations for determining the function $\chi(t)$. For the sake of brevity these equations may be written in the following form:

$$\Omega_i[\chi(t), t] = \int_0^t \psi_i[\chi(t), t, \chi(\tau), \tau]\, d\tau, \quad (i = 1, 2), \tag{2.8}$$

where we have introduced the notation

$$\Omega_1[\chi(t), t] = -2\nu_2(t) + 2\exp(-bt)\nu_{31}(t) + \sqrt{t}\left\{\frac{\partial F[\chi(t), t]}{\partial x} - \frac{\alpha\chi(t)}{a^2} - \exp(-bt)\frac{\partial\Phi_1[\chi(t), t]}{\partial x}\right\};$$

$$\Omega_2[\chi(t), t] = -2\nu_2(t) + 2\nu_{32}(t) + \sqrt{t}\left\{\frac{\partial F[\chi(t), t]}{\partial x} - \frac{\partial\Phi_2[\chi(t), t]}{\partial x} - \frac{\varepsilon}{a^2}\chi(t)\right\}.$$

Equation (2.8) may also be written in one of the following two ways:

$$\chi(t) = \nu_1(t) Q_{1i}[\chi(t), t] \tag{2.9}$$

or

$$L - \chi(t) = \nu_{4i}(t) Q_i[\chi(t), t]. \tag{2.10}$$

Here Q_{1i} and $Q_i (i = 1, 2)$ are certain nonlinear operators. Applying the method of successive approximations to (2.9) and (2.10)

$$\begin{aligned} &\chi_{m+1}(t) = \nu_1(t) Q_{1i}[\chi_m(t), t], \quad \chi_0(t) = 0; \\ &L - \chi_{m+1}(t) = \nu_{4i}(t) Q_i[\chi_m(t), t], \quad \chi_0(t) = L, \end{aligned} \tag{2.11}$$

we obtain the inequalities ($\|\chi\| = \max|\chi(t)|$)

$$\|\chi_{m+1} - \chi_m\| \leqslant q\|\chi_m - \chi_{m-1}\|. \tag{2.12}$$

The successive approximations $\chi m(t)$ converge if $q < 1$. It is clear from Eqs. (2.11), (2.12), and (1.14) that this inequality may in fact be satisfied, at least subject to certain limitations imposed upon the constants entering into the conditions of the problem and on the function $\varphi(x)$.

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